ISRAEL JOURNAL OF MATHEMATICS, Vol. 60, No. 3, 1987

LOCAL ANALYTIC INVARIANTS AND SPLITTING THEOREMS IN DIFFERENTIAL ANALYSIS

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Dedicated to the memory of David P. Milman

ABSTRACT

We show that several classical problems concerning the splitting of exact sequences of spaces of differentiable functions can be reduced to questions of semicontinuity of discrete local invariants in analytic geometry. We thus provide a uniform approach to the continuous linear solution of the division, composition and extension problems in differential analysis, recovering the classical theorems and giving many new results.

§1. Introduction

We show that several classical problems concerning the splitting of exact sequences of spaces of diferentiable functions can be reduced to questions of semicontinuity of discrete local invariants in analytic geometry. We thus provide a uniform approach to the continuous linear solution of the division, composition and extension problems in differential analysis, recovering the classical theorems and giving many new results. Our techniques come, on the functional analytic side, from splitting theorems for exact sequences of Fréchet spaces due to Vogt and Wagner [24, 25, 26], and, on the geometric side, from our work on "Relations among analytic functions" [4]. Semicontinuity of the local invariants implies the existence of natural analytic stratifications of our spaces (together with their functional structures). The continuous linear

† Research partially supported by NSERC operating grant A9070.

†† Supported by NSERC University Research Fellowship and operating grant A8849.

Received February 24, 1986

solutions to our problems are obtained, as in [5], by applying the Fréchet space techniques inductively over the strata.

The classical prototype of our main theorem is the existence of a continuous linear extension operator for $\mathscr{C}^\infty$ functions defined on a closed half-line (Mitiagin [17], Seeley [19]). Together with Whitney's theorem on differentiable even functions [27], it provides the following composition theorem: there is a continuous linear operator λ taking $\mathscr{C}^\infty$ even functions $f(x)$ into $\mathscr{C}^\infty$ functions $\lambda(f)(y)$ such that $f(x) = \lambda(f)(x^2)$. The extension theorem of Mityagin and Seeley follows also from Mather's continuous linear version of Malgrange's division theorem [12]. Conversely, a slightly more general extension theorem due to Stein [20] (cf. [2]) implies the existence of a continuous linear splitting in Malgrange's theorem [15].

These results are very special aspects of a general problem: Let M and N denote analytic manifolds (over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$) and let $\varphi : M \to N$ be an analytic mapping. Let A and B be $p \times q$ and $p \times r$ matrices, respectively, whose entries are analytic functions on M. Suppose $\mathbf{K} = \mathbf{R}$. Then φ induces a (continuous linear) homomorphism $\varphi^* : \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(M)$, where $\mathscr{C}^\infty(M)$ denotes the Fréchet algebra of $\mathscr{C}^\infty$ functions on M. Let $\Phi: \mathscr{C}^\infty(N)^q \to \mathscr{C}^\infty(M)^p$ denote the module homomorphism over φ^* defined by $\Phi(g)(x) = A(x) \cdot g(\varphi(x))$, where $g = (g_1, \ldots, g_q) \in \mathscr{C}^\infty(N)^q$, and let $B\cdot : \mathscr{C}^\infty(M)^r \to \mathscr{C}^\infty(M)^p$ denote the $\mathscr{C}^\infty(M)$-homomorphism induced by multiplication by the matrix B. (This notation will be fixed throughout the article.) We ask:

(1) Is $\Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r$ a closed subspace of $\mathscr{C}^\infty(M)^p$?

(2) If so, does the canonical surjection

$$\mathscr{C}^\infty(N)^q \otimes \mathscr{C}^\infty(M)^r \to \Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r$$

admit a continuous linear splitting?

The natural setting for our approach to these questions is in local analytic geometry: Let $\mathcal{O}_M$ and $\mathcal{O}_N$ denote the sheaves of germs of analytic functions on M and N, respectively. Then φ induces a sheaf homomorphism $\varphi^* : \mathcal{O}_N \to \mathcal{O}_M$. In analogy with the homomorphisms Φ and $B\cdot$ defined above, there is an induced homomorphism $\Phi: \mathcal{O}_N^q \to \mathcal{O}_M^p$ over $\varphi^* : \mathcal{O}_N \to \mathcal{O}_M$, and an induced $\mathcal{O}_M$-homomorphism $B : \mathcal{O}_M^r \to \mathcal{O}_M^p$. Let $\Psi : \mathcal{O}_N^q \to \operatorname{Coker} B$ denote the homomorphism over φ^* induced by Φ. (Locally, any φ^*-homomorphism from $\mathcal{O}_N^q$ to a coherent $\mathcal{O}_M$-module has this form.)

Let $a \in M$. Then φ^* determines a homomorphism of local rings $\varphi_a^* : \mathcal{O}_{N,\varphi(a)} \to \mathcal{O}_{M,a}$. We write $\mathcal{O}_a = \mathcal{O}_{M,a}$, etc., when there is no possibility of confusion. From Φ, B and Ψ, there are induced homomorphisms $\Phi_a : \mathcal{O}_{\varphi(a)}^q \to \mathcal{O}_a^p$ over φ_a^*,

$B_a : \mathcal{O}_a^r \to \mathcal{O}_a^p$ over $\mathcal{O}_a$, and $\Psi_a : \mathcal{O}_{\varphi(a)}^q \to \mathcal{O}_a^p / B_a \cdot \mathcal{O}_a^r$ over φ_a^*. Let $\hat{\varphi}_a^* : \hat{\mathcal{O}}_{\varphi(a)} \to \hat{\mathcal{O}}_a$, $\hat{\Phi}_a : \hat{\mathcal{O}}_{\varphi(a)}^q \to \hat{\mathcal{O}}_a^p$, $\hat{B}_a : \hat{\mathcal{O}}_a^r \to \hat{\mathcal{O}}_a^p$, and $\hat{\Psi}_a : \hat{\mathcal{O}}_{\varphi(a)}^q \to \operatorname{Coker} \hat{B}_a$ denote the corresponding homomorphisms of the completions. With respect to any local coordinate system $x = (x_1, \ldots, x_m)$ in a neighborhood of a, $\mathcal{O}_a$ (respectively, $\hat{\mathcal{O}}_a$) identifies with the ring of convergent (respectively, formal) power series $\mathbf{K}\{x\}$ (respectively, $\mathbf{K}[[x]]$). If $G \in \hat{\mathcal{O}}_{\varphi(a)}$, then $\hat{\varphi}_a^*(G)$ is given by composition of formal power series: with respect to local coordinates $y = (y_1, \ldots, y_n)$ near $\varphi(a)$, we substitute the formal Taylor series at a of the components of φ (without their constant terms) for the variables y_j in $G(y_1, \ldots, y_n)$. We also write $G \circ \hat{\varphi}_a$ for $\hat{\varphi}_a^*(G)$. If $\mathbf{K} = \mathbf{R}$, there is a Taylor series homomorphism $f \mapsto \hat{f}_a$ from $\mathscr{C}^\infty(M)^p$ onto $\hat{\mathcal{O}}_a^p$ (onto by the lemma of E. Borel [22, IV.3.5]).

The way that the *module of formal relations*

$$\mathscr{R}_a = \operatorname{Ker} \hat{\Psi}_a$$

varies with respect to a plays the central role in the problems above. Let H_a denote the *Hilbert–Samuel function* of $\hat{\mathcal{O}}_{\varphi(a)}^q / \mathscr{R}_a$:

$$H_a(k) = \dim_{\mathbf{K}} \frac{\hat{\mathcal{O}}_{\varphi(a)}^q}{\mathscr{R}_a + \mathfrak{M}_{\varphi(a)}^{k+1} \cdot \hat{\mathcal{O}}_{\varphi(a)}^q}$$

(Where $\mathfrak{M}_{\varphi(a)}$ denotes the maximal ideal of $\mathcal{O}_{\varphi(a)}$).

If $\mathbf{K} = \mathbf{R}$, let $(\Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r)^\wedge$ denote the elements of $\mathscr{C}^\infty(M)^p$ which *formally belong to* $\Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r$; i.e., $\{f \in \mathscr{C}^\infty(M)^p :$ for all $b \in \varphi(M)$, there exists $G_b \in \hat{\mathcal{O}}_b^q$ such that $\hat{f}_a - \hat{\Phi}_a(G_b) \in \operatorname{Im} \hat{B}_a$, for all $a \in \varphi^{-1}(B)\}$. Then $(\Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r)^\wedge$ is a closed subspace of $\mathscr{C}^\infty(M)^p$ (Proposition 3.1).

THEOREM 1.1. *Let $\mathbf{K} = \mathbf{R}$. Let M, N, φ, A and B be as above. Suppose that φ is proper. Assume that the Hilbert–Samuel function H_a is Zariski semicontinuous on M. Then*:

(1) $\Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r = (\Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r)^\wedge$; *in particular*, $\Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r$ *is a closed subspace of* $\mathscr{C}^\infty(M)^p$.

(2) *The canonical surjection*

$$\mathscr{C}^\infty(N)^q \oplus \mathscr{C}^\infty(M)^r \to \Phi \mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r$$

admits a continuous linear splitting.

Assertion (1) of this theorem is proved in [4]. The functional analytic statement (2) will be established here.

Zariski semicontinuity of the Hilbert–Samuel function H_a means: For every irreducible (germ of a) closed analytic subset X of M, there is a proper closed analytic subset Y of X such that (1) if $a, b \in X - Y$, then $H_a(k) = H_b(k)$ for all k, and (2) if $a \in X - Y$, $b \in Y$, then $H_a(k) \leqq H_b(k)$ for all k and $H_a(k) < H_b(k)$ for some k. Thus, given $a \in M$, $\{b \in M : H_a(k) \leqq H_b(k) \text{ for all } k\}$ is a closed analytic subset of M.

We conjecture that the Hilbert–Samuel function H_a is Zariski semicontinuous, in general. We have proved it in each of the following cases [4, Thm. C]:

(a) In the algebraic category. (Here the Hilbert–Samuel function is upper semicontinuous with respect to the (algebraic) Zariski topology.)

(b) If $\Psi = \varphi^* : \mathcal{O}_N \to \mathcal{O}_M$ and φ is *regular* in the sense of Gabrielov [7]; i.e., the Krull dimension of $\mathcal{O}_{\varphi(a)}/\mathrm{Ker}\ \varphi_a^*$ is locally constant on M.

(c) If φ is *locally finite*; i.e., for all $a \in M$, $\mathcal{O}_a$ is a finite $\mathcal{O}_{\varphi(a)}$-module via the homomorphism φ_a^*.

Case (c) includes the coherent case ($M = N$, φ = identity), in which semicontinuity is known classically. The modules of formal relations $\mathscr{R}_a$, $a \in M$, do not, in general, form a coherent object. Nor, in general, is the Hilbert–Samuel function of $\hat{\mathcal{O}}^a_{\varphi(a)}/\mathscr{R}_{*\varphi(a)}$, where $\mathscr{R}_{*b} = \bigcap_{a \in \varphi^{-1}(b)} \mathrm{Ker}\ \hat{\Psi}_a$, semicontinuous as a function of $a \in M$ (cf. [4, Rmk. 2.3]).

Special cases of Theorem 1.1 include:

Division theorems. $A \cdot \mathscr{C}^\infty(M)^q$ is closed in $\mathscr{C}^\infty(M)^p$ (Malgrange [11, VI.1]). Moreover, the canonical surjection $\mathscr{C}^\infty(M)^q \to A \cdot \mathscr{C}^\infty(M)^q$ admits a continuous linear spliting [5, Thm. 0.1.1]. Note that $A \cdot \mathscr{C}^\infty(M)^q$ does not, in general, have a closed linear complement in $\mathscr{C}^\infty(M)^p$: e.g., $M = \mathbf{R}^2$, $p = q = 1$, $A = x_1^2 + x_2^2$ [5].

The Malgrange–Mather division theorem [12]: Let $P(t, \lambda)$ denote the polynomial $t^d + \Sigma_{j=1}^d \lambda_j t^{d-j}$ with generic coefficients $\lambda = (\lambda_1, \ldots, \lambda_d)$. Then every $\mathscr{C}^\infty$ function $f(x, t) = f(x_1, \ldots, x_n, t)$ can be written

$$f(x, t) = P(t, \lambda) \cdot q(x, t, \lambda) + \sum_{j=1}^{d} g_j(x, \lambda) t^{d-j},$$

where $q, g_1, \ldots, g_d$ are $\mathscr{C}^\infty$ and depend in a continuous linear way on f. This is equivalent to the conclusion of Theorem 1.1 in the case that $M = N = \mathbf{R}^{n+d}$, $\varphi : M \to N$ is the mapping

$$\varphi(x, t, \lambda_1, \ldots, \lambda_{d-1}) = \left(x, \lambda_1, \ldots, \lambda_{d-1}, -t^d - \sum_{j=1}^{d-1} \lambda_j t^{d-j}\right)$$

where $x=(x_1,\dots,x_n)$, $B=0$ and $A(x,t,\lambda_1,\dots,\lambda_{d-1})$ is the $1\times d$ matrix $(t^{d-1}t^{d-2}\cdots 1)$. Indeed, given $f(x,t)$ $\mathscr{C}^\infty$, there exists $g(x,\lambda)\in\mathscr{C}^\infty(\mathbf{R}^{n+d})^d$, $g=(g_1,\dots,g_d)$, such that $f=A\cdot(g\circ\varphi)$ if and only if $f(x,t)-\Sigma_{j=1}^d t^{d-j}g_j(x,\lambda)$ divided by $t^d+\Sigma_{j=1}^{d-1}\lambda_j t^{d-j}+\lambda_d=P(t,\lambda)$ is a $\mathscr{C}^\infty$ function. Theorem 1.1 applies in this case because f belongs to $(\Phi\mathscr{C}^\infty(\mathbf{R}^{n+d})^d)^\wedge$, by the formal Weierstrass division theorem (cf. [4, Example 4.5]).

Composition theorems. There is a necessary condition for $\varphi^*\mathscr{C}^\infty(N)$ to be closed: φ must be *semiproper*; i.e., for every compact $L\subset N$, there is a compact $K\subset M$ such that $\varphi(K)=L\cap\varphi(M)$ [5, Prop. 1.4.1]. The composition theorem of [3] asserts that $\varphi^*\mathscr{C}^\infty(N)$ is closed if φ is semiproper and $\varphi(M)$ is *Nash subanalytic* (i.e., the image of a proper real analytic mapping which is regular as in (b) above, so that Theorem 1.1(1) applies). Special cases of this theorem include Glaeser's composition theorem [8], Schwarz's and Luna's theorems on differentiable invariants [18, 10] and a result of Tougeron [23].

If φ is semiproper and $\varphi(M)$ is Nash subanalytic, then the surjection $\mathscr{C}^\infty(N)\to\varphi^*\mathscr{C}^\infty(N)$ admits a continuous linear splitting, by Theorem 1.1(2). The splitting provides an extension operator for $\mathscr{C}^\infty$ functions on $\varphi(M)$: see below. In the case of diferentiable invariants, the existence of a splitting is due to Mather [14]. Here φ is given by a set of generators for the algebra of invariant polynomials on a linear representation of a reductive real algebraic group. If the group is compact, then averaging, together with Schwarz's theorem, shows that $\varphi^*\mathscr{C}^\infty(N)$ has a closed linear complement in $\mathscr{C}^\infty(M)$. In general, it is not true that $\varphi^*\mathscr{C}^\infty(N)$ has a closed linear complement: e.g., $M=N=\mathbf{R}$, $\varphi(x)=x^k$, where $k\geqq 3$ [16].

Extension theorems. Let Y be a closed subset of N. Denote by $\mathscr{C}^\infty(Y)$ the space of restrictions to Y of elements of $\mathscr{C}^\infty(N)$, and by $\mathscr{E}(Y)$ the space of $\mathscr{C}^\infty$ Whitney fields on Y (see Definition 3.5). Whitney's extension theorem [11, I.4.1] asserts that the canonical restriction mapping $\mathscr{E}(N)\cong\mathscr{C}^\infty(N)\to\mathscr{E}(Y)$ is onto. (Its kernel is the ideal $\mathscr{E}(N;Y)$ of $\mathscr{C}^\infty$ functions which are flat on Y.)

Grothendieck pointed out that the restriction (Taylor series homomorphism) $\mathscr{E}(\mathbf{R}^n)\to\mathscr{E}(\text{point})$ does not admit a continuous linear splitting [22, IV.3.9]. Suppose $Y=\overline{\operatorname{int} Y}$ (so that $\mathscr{E}(Y)=\mathscr{C}^\infty(Y)$). If Y is subanalytic, then there is a continuous linear extension operator $E:\mathscr{E}(Y)\to\mathscr{E}(N)$ [1]; i.e., the restriction mapping $\mathscr{E}(N)\to\mathscr{E}(Y)$ splits. This is false in general: e.g., $N=\mathbf{R}^2$, $Y=\{(y_1,y_2):y_1\geqq 0,\ 0\leqq y_2\leqq e^{-1/y_1}\}$ (Tidten [21]).

Suppose Y is subanalytic. Clearly, if $Y=\overline{\operatorname{int} Y}$, then Y is Nash. In general, if Y is Nash, then there is a continuous linear extension operator $E:\mathscr{C}^\infty(Y)\to$

$\mathscr{C}^\infty(N)$ [5, Thm. 0.2.1]. Let $\varphi : M \to N$ be a proper analytic mapping such that $\varphi(M) = Y$.[†] Then φ^* induces an injection $\mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(M)$ with image $\varphi^*\mathscr{C}^\infty(N)$. Therefore, if $\varphi^*\mathscr{C}^\infty(N)$ is a closed subspace of $\mathscr{C}^\infty(M)$, the existence of an extension operator is equivalent to the existence of a continuous linear splitting for the surjection $\varphi^* : \mathscr{C}^\infty(N) \to \varphi^*\mathscr{C}^\infty(N)$. Thus Theorem 1.1 applies also to the extension problem.

Suppose that φ is regular. The approach of [5] and of this article is as follows: We can assume that Y and thus M is compact. Then $\mathscr{C}^\infty(M)$ is isomorphic, as a Fréchet space, to the space s of rapidly decreasing sequences of real or complex numbers [5, Prop. 6.3]; cf. Proposition 2.4 below. There is an exact sequence

$$0 \to \mathscr{C}^\infty(N; Y) \to \mathscr{C}^\infty(N) \to \mathscr{C}^\infty(Y) \to 0,$$

where $\mathscr{C}^\infty(N; Y)$ denotes the ideal in $\mathscr{C}^\infty(N)$ of functions which vanish on Y. By the composition theorem [3], $\mathscr{C}^\infty(Y)$ is a closed subspace of s. Since Y is Nash subanalytic, $\mathscr{C}^\infty(N; Y)$ is a quotient of s [5, Thm. 6.6]. (In the case that $Y = \overline{\operatorname{int} Y}$, this is a result of Tidten [21]; cf. Theorem 2.5.) Therefore, the sequence splits, by the theorem of Vogt and Wagner [25] (see Section 2 below).

In the general context of Theorem 1.1, let $\mathscr{R}(N)$ denote the kernel of the surjection

$$\mathscr{C}^\infty(N)^q \to \frac{\Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r}{B \cdot \mathscr{C}^\infty(M)^r} \tag{1.2}$$

induced by Φ. In order to apply the Vogt–Wagner results to prove Theorem 1.1(2), we need:

(i) $\Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r$ is a closed subspace of $\mathscr{C}^\infty(M)^p$ (Theorem 1.1(1)), and thus a closed subspace of s in the case that M is compact.

(ii) $\mathscr{R}(N)$ is a quotient of s.

The main ingredient in our proof of (ii) is a natural analytic stratification associated to the modules of formal relations $\mathscr{R}_a$:

Let $s \in \mathbf{N}$. Let M_φ^s denote the s-fold fiber product

$$M_\varphi^s = \{\mathbf{a} = (a^1, \ldots, a^s) \in M^s : \varphi(a^1) = \cdots = \varphi(a^s)\},$$

and let $\boldsymbol{\varphi} : M_\varphi^s \to N$ denote the induced mapping. If $\mathbf{a} = (a^1, \ldots, a^s) \in M_\varphi^s$, put

[†] Such a φ always exists. For a simple proof see E. Bierstone and P. D. Milman, *Semianalytic and subanalytic sets*, Thm. 5.1 (to appear).

$$\mathcal{R}_{\mathbf{a}} = \bigcap_{i=1}^{s} \mathcal{R}_{a^i},$$

and let $H_{\mathbf{a}}(k)$ denote the Hilbert–Samuel function of $\hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}/\mathcal{R}_{\mathbf{a}}$. Zariski semicontinuity of $H_{\mathbf{a}}$, for a given positive integer s, is equivalent to Zariski semicontinuity of H_a (i.e., of $H_{\mathbf{a}}$ in the case $s = 1$) [4, Prop. 9.6].

If φ is proper, then, locally in N, there is a bound s on the number of distinct submodules $\mathcal{R}_a$ of $\hat{\mathcal{O}}^q_b$, where $a \in \varphi^{-1}(b)$ [4, Cor. 11.6]. We work locally and fix such an s. Thus, for all $b \in \varphi(M)$, there exists $\mathbf{a} \in \varphi^{-1}(b)$ such that $\mathcal{R}_{\mathbf{a}} = \bigcap_{a\in\varphi^{-1}(b)} \mathcal{R}_a$.

Suppose X is a locally closed subset of M^s_φ. We say that a function h on X is *meromorphic* if each point of $\bar{X}$ admits a neighborhood in which there are finitely many pairs of analytic functions f_i, g_i such that $X \cap \bigcap_i g_i^{-1}(0) = \varnothing$ and $h \cdot g_i = f_i$ on X, for each i. Let $\mathcal{M}(X)$ denote the ring of meromorphic functions on X. If $\mathbf{K} = \mathbf{C}$, then a function on X is meromorphic if and only if its germ at each $\mathbf{a} \in \bar{X}$ is induced by a meromorphic function (in the classical sense) defined on an analytic set containing $\bar{X}$ near $\mathbf{a}$, whose poles lie outside X.

Denote by $\mathcal{M}(X)[[y]]$ the ring of formal power series in $y = (y_1, \ldots, y_n)$ with coefficients in $\mathcal{M}(X)$. If $\mathbf{a} \in X$, there is a mapping $\mathcal{M}(X)[[y]] \to \mathbf{K}[[y]]$ defined by evaluating coefficients at $\mathbf{a}$. If N is an open subset of $\mathbf{K}^n$ and $b \in N$, we identify $\hat{\mathcal{O}}_b$ with the formal power series ring $\mathbf{K}[[y]]$. Suppose $G \in \hat{\mathcal{O}}^q_b$; say $G = (G_1, \ldots, G_q)$, where $G_j(y) = \sum_{\beta\in\mathbf{N}^n} g_{\beta,j} y^\beta$ (in multiindex notation), $j = 1, \ldots, q$. We write $G = \sum_{\beta,j} g_{\beta,j} y^{\beta,j}$, where $y^{\beta,j}$ denotes the q-tuple with y^β in the j'th place and zeros elsewhere.

THEOREM 1.3. *Suppose that φ is proper, that N is an open subset of $\mathbf{K}^n$, and that s is as above. Assume that the Hilbert–Samuel function $H_{\mathbf{a}}$ is Zariski semicontinuous on M^s_φ. Then there is a locally finite partition $\{X_\mu\}_{\mu\in\mathbf{N}}$ of M^s_φ such that, for each μ:*

(1) *X_μ is a relatively compact connected smooth semianalytic subset of M^s_φ.*

(2) *$\bar{X}_\mu - X_\mu \subset \bigcup_{\lambda<\mu} X_\lambda$.*

(3) *$\boldsymbol{\varphi} | X_\mu$ has constant rank.*

(4) *Let $Y_\mu = \boldsymbol{\varphi}(\bigcup_{\lambda\le\mu} X_\lambda)$. Then, for all $\mathbf{a} \in X_\mu - \boldsymbol{\varphi}^{-1}(Y_{\mu-1})$, $\mathcal{R}_{\mathbf{a}} = \bigcap_{a\in\varphi^{-1}(\boldsymbol{\varphi}(\mathbf{a}))} \mathcal{R}_a$.*

(5) *There exist finitely many elements $G^i_\cdot(y) \in \mathcal{M}(X_\mu)[[y]]^q$ with the following properties: Write $G^i_\cdot(y) = \sum_{\beta,j} g^i_{\beta,j} y^{\beta,j}$ where each $g^i_{\beta,j} \in \mathcal{M}(X_\mu)$. If $\mathbf{a} \in X_\mu$, put $G^i_{\mathbf{a}}(y) = \sum g^i_{\beta,j}(\mathbf{a}) y^{\beta,j}$. Then:*

(5.1) *For each $\mathbf{a} \in X_\mu$, the $G^i_{\mathbf{a}}(y)$ generate $\mathcal{R}_{\mathbf{a}} \subset \hat{\mathcal{O}}^q_{\boldsymbol{\varphi}(\mathbf{a})} = \mathbf{K}[[y]]^q$.*

(5.2) *If* $\mathbf{a}\in X_\mu - \varphi^{-1}(Y_{\mu-1})$, *then each* $g^i_{\beta,j}(\mathbf{a})$ *depends only on* $b=\varphi(\mathbf{a})$. *Write* $g^i_{\beta,j}(\mathbf{a}) = g^i_{\beta,j}(b)$.

(5.3) *For each* i, *let* G^i *denote the field of formal power series* $G^i_b(y) = \Sigma\, g^i_{\beta,j}(b)y^{\beta,j}$, $b\in Y_\mu - Y_{\mu-1}$. *Then* G^i *is analytic with respect to the variables of the immersed submanifold* $Y_\mu - Y_{\mu-1}$ *of* $N - Y_{\mu-1}$, *and formal with respect to the variables in the normal direction. (See Definition and Remarks* 6.8.) *In particular, if* $\mathbf{K}=\mathbf{R}$, *then* $G^i \in \mathscr{E}(Y_\mu - Y_{\mu-1})^q$.

We actually have a more precise version of Theorem 1.3 (see (6.1)–(6.6) and Proposition 6.10 in Section 6): Zariski semicontinuity of the Hilbert–Samuel function $H_\mathbf{a}$ is equivalent to Zariski semicontinuity of a "diagram of initial exponents" $\mathfrak{N}_\mathbf{a}$ associated to the module $\mathscr{R}_\mathbf{a}$ [4, Thm. A] (see Section 4 below). The diagram $\mathfrak{N}_\mathbf{a}$ gives a combinatorial picture of $\mathscr{R}_\mathbf{a}$, in the spirit of the classical Newton diagram of a formal power series. The stratification $\{X_\mu\}$ can be chosen so that $\mathfrak{N}_\mathbf{a}$ is constant on each X_μ, and the $G^i_\mathbf{a}(y)$ are "special generators" of $\mathscr{R}_\mathbf{a}$, uniquely determined by the diagram $\mathfrak{N}_\mathbf{a}$ (cf. [4, Thm. B]), so that (5.2) follows from (4).

Let $Z\subset Y$ be closed subsets of N. Then the space $\mathscr{E}(Y;Z)$ of Whitney fields on Y which are flat on Z is a quotient of s, by Tidten's theorem (Theorem 2.5). Define $\mathscr{R}(Y;Z)$ as $\{G\in\mathscr{E}(Y;Z)^q : \hat{G}_b \in \bigcap_{a\in\varphi^{-1}(b)} \mathscr{R}_a$, for all $b\in(Y-Z)\cap\varphi(M)\}$ (cf. Definitions 3.5 and 3.9). Put $\mathscr{R}(Y)=\mathscr{R}(Y;\varnothing)$. ($\mathscr{R}(N)$ is the kernel of (1.2), in agreement with the earlier notation.) Using (4) and (5) of Theorem 1.3, we will prove that each $\mathscr{R}(Y_\mu; Y_{\mu-1})$ is a quotient of s (Theorem 6.7). On the other hand, it follows from Theorem 1.1(1) that, for each μ, the sequence

$$0\to \mathscr{R}(Y_\mu; Y_{\mu-1}) \to \mathscr{R}(Y_\mu) \to \mathscr{R}(Y_{\mu-1}) \to 0$$

is exact (Corollary 3.11). Theorem 1.1(2) follows by induction on μ, using the theorem of Vogt and Wagner (see Section 3).

§2. The theorems of Vogt, Wagner and Tidten

Let s be the space of *rapidly decreasing sequences* of real or complex numbers; i.e., sequences $x=(x_j)_{i\in\mathbf{Z}}$ such that, for every $k\in\mathbf{N}$,

$$\|x\|_k = \sup_j |j|^k |x_j| < \infty.$$

With the seminorms $\|\cdot\|_k$, s has the structure of a nuclear Fréchet space. The Fourier transform induces an isomorphism $\mathscr{C}^\infty(S^1)\cong s$.

PROPOSITION 2.1 [25]. (1) *The direct sum of a finite number of copies of s is isomorphic to s.*

(2) $s^N = \Pi_{k\in\mathbf{N}}\, s$ *is a quotient of s.*

THEOREM 2.2 [25, 26]. *Let*

$$\begin{array}{ccccccccc} & & & & & & H & & \\ & & & & & & \downarrow{\scriptstyle \pi} & & \\ 0 & \to & E & \to & F & \xrightarrow{\mu} & G & \to & 0 \end{array}$$

be a diagram of Fréchet spaces and continuous linear mappings, where the horizontal sequence is exact. Suppose that E is a quotient of s and that H is a closed subspace of s. Then there is a continuous linear mapping $\lambda : H \to F$ *such that* $\mu \circ \lambda = \pi$.

COROLLARY 2.3 [25, 26]. *Let*

$$0 \to E \to F \to G \to 0$$

be an exact sequence of Fréchet spaces and continuous linear mappings. If E and G are quotients of s, then so is F.

PROPOSITION 2.4 [5, Prop. 6.3]. *Let M be a* $\mathscr{C}^\infty$ *manifold (with or without boundary). Then:*

(1) $\mathscr{C}^\infty(M)$ *is a quotient of s.*

(2) *If M is compact, then* $\mathscr{C}^\infty(M) \cong s$.

THEOREM 2.5 [21; 5, Thm. B.1]. *Let M be a* $\mathscr{C}^\infty$ *manifold and let X be a closed subset of M. Then the space* $\mathscr{E}(M; X)$ *of* $\mathscr{C}^\infty$ *functions which are flat on X is a quotient of s.*

§3. The main theorem

We will formulate a more general, relative version of Theorem 1.1 (Theorem 3.6 below). This is needed, in any case, to deduce 1.1(2) from the special case that M is compact, using a partition of unity. The notation of the introduction will be used in this section and throughout the paper. Let $\mathbf{K} = \mathbf{R}$.

PROPOSITION 3.1. $(\Phi\mathscr{C}^\infty(N)^q + B \cdot \mathscr{C}^\infty(M)^r)^\wedge$ *is a closed subspace of* $\mathscr{C}^\infty(M)^p$.

Our proof of Proposition 3.1 is based on three lemmas which follow. If $a \in M$ and $k \in \mathbf{N}$, let $J_a^k(M)$ denote the finite-dimensional vector space

$\hat{\mathcal{O}}_a/\mathfrak{M}_a^{k+1}\cdot\hat{\mathcal{O}}_a$. Then $\hat{\Phi}_a:\hat{\mathcal{O}}^q_{\varphi(a)}\to\hat{\mathcal{O}}^p_a$ and $\hat{B}_a:\hat{\mathcal{O}}^r_a\to\hat{\mathcal{O}}^p_a$ induce linear mappings $\Phi^k_a: J^k_{\varphi(a)}(N)^q\to J^k_a(M)^p$ and $B^k_a: J^k_a(M)^r\to J^k_a(M)^p$, respectively; Φ^k_a induces $\Psi^k_a: J^k_{\varphi(a)}(N)^q\to J^k_a(M)^p/\mathrm{Im}\, B^k_a$. If $F\in\hat{\mathcal{O}}^p_a$ (respectively, $f\in\mathscr{C}^\infty(M)^p$), let F^k (respectively, f^k_a) denote the image of F (respectively, f) in $J^k_a(M)^p$.

Let $b\in\varphi(M)$. If $k\in\mathbf{N}$ and $X\subset\varphi^{-1}(b)$, put $\mathscr{F}^k_X=\{f\in\mathscr{C}^\infty(M)^p$: there exists $G\in\hat{\mathcal{O}}^q_b$ such that $\hat{f}_a-\hat{\Phi}_a(G)\in\mathrm{Im}\,\hat{B}_a+\mathfrak{M}_a^{k+1}\cdot\hat{\mathcal{O}}^p_a$, for all $a\in X\}$ and let $\mathscr{R}^k_X=\bigcap_{a\in X}\mathrm{Ker}\,\Psi^k_a$.

Lemma 3.2. *$\mathscr{F}^k_{\varphi^{-1}(b)}=\bigcap\{\mathscr{F}^k_X: X\subset\varphi^{-1}(b)$ and X is finite$\}$.*

Proof. If $X_1\subset X_2\subset\varphi^{-1}(b)$, then $\mathscr{R}^k_{X_2}\subset\mathscr{R}^k_{X_1}$. It follows that, for all $k\in\mathbf{N}$, there is a finite subset X^k of $\varphi^{-1}(b)$ such that $\mathscr{R}^k_{\varphi^{-1}(b)}=\mathscr{R}^k_{X^k}$.

Suppose $f\in\bigcap\{\mathscr{F}^k_X: X\subset\varphi^{-1}(b)$ and X is finite$\}$. Let $G\in\mathcal{O}^q_b$ such that, for all $a\in X^k$, $f^k_a-\Phi^k_a(G^k)\in\mathrm{Im}\,B^k_a$. Suppose $X\supset X^k$ is finite. Let $H\in\mathcal{O}^q_b$ such that $f^k_a-\Phi^k_a(H^k)\in\mathrm{Im}\,B^k_a$, for all $a\in X$. Then $(G-H)^k\in\mathscr{R}^k_{X^k}=\mathscr{R}^k_{\varphi^{-1}(b)}$, so that $f^k_a-\Phi^k_a(G^k)\in\mathrm{Im}\,B^k_a$, for all $a\in X$. The result follows. □

Lemma 3.3 (Chevalley estimate; cf. [4, Lemma 8.2.2]). *There is a function $l=l(k,b)$ from $\mathbf{N}$ to itself such that, if $G\in\hat{\mathcal{O}}^q_b$ and $\hat{\Phi}_a(G)\in\mathrm{Im}\,\hat{B}_a+\mathfrak{M}_a^{l+1}\cdot\hat{\mathcal{O}}^p_a$ for all $a\in\varphi^{-1}(b)$, then $G\in\bigcap_{a\in\varphi^{-1}(b)}\mathrm{Ker}\,\hat{\Psi}_a+\mathfrak{M}_b^{k+1}\cdot\hat{\mathcal{O}}^q_b$.*

Lemma 3.4. *$\bigcap_{k=1}^\infty\mathscr{F}^k_{\varphi^{-1}(b)}=\{f\in\mathscr{C}^\infty(M)^p$: there exists $G\in\hat{\mathcal{O}}^q_b$ such that $\hat{f}_a-\hat{\Phi}_a(G)\in\mathrm{Im}\,\hat{B}_a$, for all $a\in\varphi^{-1}(b)\}$.*

Proof. Put $k_0=0$. Let $k_{s+1}=\max(k_s+1,l_s)$, where $l_s=l(k_s,b)$, $s=0,1,2,\ldots$. Suppose $f\in\bigcap_{k=1}^\infty\mathscr{F}^k_{\varphi^{-1}(b)}$. We define $G_s\in\hat{\mathcal{O}}^q_b$ such that $\hat{f}_a-\hat{\Phi}_a(G_s)\in\mathrm{Im}\,\hat{B}_a+\mathfrak{M}_a^{l_s+1}\cdot\hat{\mathcal{O}}^p_a$, for all $a\in\varphi^{-1}(b)$, and $G_{s+1}-G_s\in\mathfrak{M}_b^{k_s+1}\cdot\hat{\mathcal{O}}^q_b$, inductively as follows:

Given G_s, choose any $H\in\hat{\mathcal{O}}^q_b$ such that $\hat{f}_a-\hat{\Phi}_a(H)\in\mathrm{Im}\,\hat{B}_a+\mathfrak{M}_a^{l_{s+1}+1}\cdot\hat{\mathcal{O}}^p_a$, for all $a\in\varphi^{-1}(b)$. Then $G_s-H\in\bigcap_{a\in\varphi^{-1}(b)}\mathrm{Ker}\,\hat{\Psi}_a+\mathfrak{M}_b^{k_s+1}\cdot\hat{\mathcal{O}}^q_b$, by Lemma 3.3. Thus there exists $G_{s+1}\in\hat{\mathcal{O}}^q_b$ such that $\hat{f}_a-\hat{\Phi}_a(G_{s+1})\in\mathrm{Im}\,\hat{B}_a+\mathfrak{M}_a^{l_{s+1}+1}\cdot\hat{\mathcal{O}}^p_a$, for all $a\in\varphi^{-1}(b)$, and $G_{s+1}-G_s\in\mathfrak{M}_b^{k_s+1}\cdot\hat{\mathcal{O}}^q_b$.

Set $G=\lim_s G_s$ (limit in the Krull topology). Then $\hat{f}_a-\hat{\Phi}_a(G)\in\bigcap_l(\mathrm{Im}\,\hat{B}_a+\mathfrak{M}_a^{l+1}\cdot\hat{\mathcal{O}}^p_a)=\mathrm{Im}\,\hat{B}_a$, for all $a\in\varphi^{-1}(b)$. □

Proof of Proposition 3.1. Let $b\in\varphi(M)$ and suppose that X is a finite subset of $\varphi^{-1}(b)$; say $X=\{a^1,\ldots,a^s\}$. Let $\mathscr{X}: J^k_b(N)^q\times\bigoplus_{i=1}^s J^k_{a^i}(M)^r\to\bigoplus_{i=1}^s J^k_{a^i}(M)^p$ denote the linear mapping $\mathscr{X}(G,H^1,\ldots,H^s)=(\Phi^k_{a^i}(G)+B^k_{a^i}(H^i))_{1\le i\le s}$, where $G\in J^k_b(N)^q$ and $H^i\in J^k_{a^i}(M)^r$, $i=1,\ldots,s$. Let $f\in\mathscr{C}^\infty(M)^p$. Then $f\in\mathscr{F}^k_X$ if and only if $(f^k_{a^1},\ldots,f^k_{a^s})\in\mathrm{Im}\,\mathscr{X}$. Therefore $\mathscr{F}^k_X$ is a closed subspace of $\mathscr{C}^\infty(M)^p$.

It follows from Lemma 3.2 and 3.4 that

$$(\Phi\mathscr{C}^\infty(N)^q + B\cdot\mathscr{C}^\infty(M)^r)^\wedge = \bigcap_{b\in\varphi(M)} \bigcap_{k=1}^{\infty} \mathscr{F}^k_{\varphi^{-1}(b)}$$

is a closed subspace of $\mathscr{C}^\infty(M)^p$. □

Definition and Remarks 3.5. Let Y be a locally closed subset of N, and let Z be a closed subset of Y (where Y has the induced topology). Let V be any open subset of N such that Y is closed in V. Then Z is closed in V. Put

$$\mathscr{E}(V;Z) = \{g\in\mathscr{C}^\infty(V) : \hat{g}_b = 0, \quad \text{for all } b\in Z\}.$$

Let $\mathscr{E}(Y;Z)$ denote the Fréchet algebra $\mathscr{E}(V;Z)/\mathscr{E}(V;Y)$. (The definition is independent of V.) If N is an open submanifold of $\mathbf{R}^n$, $\mathscr{E}(Y;Z)$ identifies with the Fréchet algebra of $\mathscr{C}^\infty$ Whitney fields on Y which are flat on Z (by Whitney's extension theorem [11, I.4.1]). If Y' is a closed subset of Y, there is a restriction mapping $G\mapsto G\mid Y'$ from $\mathscr{E}(Y;Z)^q$ onto $\mathscr{E}(Y';Y'\cap Z)^q$. Put $\mathscr{E}(Y) = \mathscr{E}(Y;\varnothing)$. If $b\in Y$, then $\mathscr{E}(\{b\}) = \hat{\mathscr{O}}_b$, and we write $G\mapsto \hat{G}_b$ for the restriction $\mathscr{E}(Y)^q\to\hat{\mathscr{O}}_b^q$ (Taylor series homomorphism). Of course, $\mathscr{E}(V) = \mathscr{C}^\infty(V)$ if V is open subset of N.

Theorem 3.6. *Suppose that φ is proper. Assume that the Hilbert–Samuel function H_a is Zariski semicontinuous on M. Let Y be a closed subanalytic subset of N. Then:*

(1) $\Phi\mathscr{E}(N;Y)^q + B\cdot\mathscr{E}(M)^r = \{f\in(\Phi\mathscr{E}(N)^q + B\cdot\mathscr{E}(M)^r)^\wedge : \hat{f}_a\in \operatorname{Im}\hat{B}_a$ *for all* $a\in\varphi^{-1}(Y)\}$. *In particular,* $\Phi\mathscr{E}(N;Y)^q + B\cdot\mathscr{E}(M)^r$ *is a closed subspace of* $\mathscr{E}(M)^p$.

(2) *The canonical surjection*

$$\mathscr{E}(N;Y)^q\oplus\mathscr{E}(M)^r\to\Phi\mathscr{E}(N;Y)^q + B\cdot\mathscr{E}(M)^r$$

admits a continuous linear splitting.

Theorem 3.6(1) has been proved in [4]. We will use (1) to prove the second assertion here. We first note two corollaries:

Corollary 3.7. *Let X be a closed subanalytic subset of M. Then:*

(1) $A\cdot\mathscr{E}(M;X)^q$ *is a closed subspace of* $\mathscr{E}(M;X)^p$.

(2) *The canonical surjection* $\mathscr{E}(M;X)^q\to A\cdot\mathscr{E}(M;X)^q$ *admits a continuous linear splitting.*

PROOF. Each assertion is the corresponding assertion of Theorem 3.6 in the special case that $\varphi=$ identity, $B=0$; the Hilbert–Samuel function is Zariski semicontinuous according to case (c) in Section 1. □

COROLLARY 3.8. *Let the hypotheses be as in Theorem* 3.6. *Then*:
(1) $\Phi\mathscr{E}(N;Y)^q+B\cdot\mathscr{E}(M;\varphi^{-1}(Y))^r$ *is a closed subspace of* $\mathscr{E}(M;\varphi^{-1}(Y))^p$.
(2) *The canonical surjection*

$$\mathscr{E}(N;Y)^q\oplus\mathscr{E}(M;\varphi^{-1}(Y))^r\to\Phi\mathscr{E}(N;Y)^q+B\cdot\mathscr{E}(M;\varphi^{-1}(Y))^r$$

admits a continuous linear splitting.

PROOF. Each assertion follows from the corresponding assertions of Theorem 3.6 and Corollary 3.7. □

DEFINITION AND REMARK 3.9. Let $Z\subset Y$ denote closed subsets of N. Define $\mathscr{R}(Y;Z)$ as $\{\mathscr{G}\in\mathscr{E}(Y;Z)^q:\hat{G}_b\in\bigcap_{a\in\varphi^{-1}(b)}\mathscr{R}_a$, for all $b\in(Y-Z)\cap\varphi(M)\}$. Then $\mathscr{R}(Y;Z)$ is a closed submodule of $\mathscr{E}(Y;Z)^q$. Put $\mathscr{R}(Y)=\mathscr{R}(Y;\varnothing)$.

PROPOSITION 3.10. *Let the hypotheses be as in Theorem* 3.6. *Let Z be a closed subanalytic subset of N. Then the restriction mapping $\mathscr{R}(N;Z)\to\mathscr{R}(Y;Y\cap Z)$ is surjective.*

PROOF. Let $G\in\mathscr{R}(Y;Y\cap Z)$. Since Y and Z are regularly situated (cf. [11, I.5] and [9, §9]), there exists $g'\in\mathscr{E}(N;Z)^q$ such that $g'\mid Y=G$. Then $f=\Phi(g')\in(\Phi\mathscr{E}(N)^q+B\cdot\mathscr{E}(M)^r)^\wedge$ and $\hat{f}_a\in\operatorname{Im}\hat{B}_a$ for all $a\in\varphi^{-1}(Y\cup Z)$. By Theorem 3.6(1), there exist $g''\in\mathscr{E}(N;Y\cup Z)^q$ and $h\in\mathscr{E}(M)^r$ such that $f=\Phi(g'')+B\cdot h$. Let $g=g'-g''$. Then $g\mid Y=G$. Moreover, $g\in\mathscr{E}(N;Z)^q$ and $\Phi(g)\in B\cdot\mathscr{E}(M)^r$, so that $g\in\mathscr{R}(N;Z)$. □

COROLLARY 3.11. *Let the hypotheses be as in Theorem* 3.6. *If Z_1, Z_2 are closed subanalytic subsets of Y, then the following sequence is exact*:

$$0\to\mathscr{R}(Y;Z_1\cup Z_2)\to\mathscr{R}(Y;Z_2)\to\mathscr{R}(Z_1;Z_1\cap Z_2)\to 0.$$

THEOREM 3.12. *Let the hypotheses be as in Theorem* 3.6. *Then $\mathscr{R}(N;Y)$ is a quotient of s.*

Theorem 3.12 will be proved in Section 6 below.

PROOF OF THEOREM 3.6(2). Let V denote a relatively compact open subanalytic subset of N. Put $X=M-\varphi^{-1}(V)$ and $Z=Y\cup(N-V)$. By Theorem 3.6(1) (and Corollary 3.7(1)), $\Phi\mathscr{E}(N;Z)^q+B\cdot\mathscr{E}(M;X)^r$ is a closed subspace of $\mathscr{E}(M; X)^p$. By a partition of unity argument, it is enough to show that the canonical surjection

$$\mathscr{E}(N; Z)^q \oplus \mathscr{E}(M; X)^r \to \Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r \tag{3.13}$$

has a continuous linear splitting.

Now Φ induces a continuous linear surjection

$$\Psi : \mathscr{E}(N; Z)^q \to \frac{\Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r}{B \cdot \mathscr{E}(M; X)^r}.$$

Clearly, $\operatorname{Ker}\Psi = \mathscr{R}(N; Z)$ (for example, by Whitney's spectral theorem [11, II.1.7] or by Corollary 3.7(1)), so that $\operatorname{Ker}\Psi$ is a quotient of s, by Theorem 3.12.

Let L denote a compact $\mathscr{C}^\infty$ submanifold of M with boundary, such that $\varphi^{-1}(V) \subset L$. Then $\Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r$ is a closed subspace of $\mathscr{E}(M; X)^p$, hence of $\mathscr{E}(L)^p \cong s$.

Consider the special case that $M = N$, $\varphi =$ identity, $B = 0$ and $Y = \varnothing$: Then $\operatorname{Ker}\Psi = \operatorname{Ker} A\,\cdot$ is a quotient of s, and $\Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r = A \cdot \mathscr{E}(M; X)^q$ is a closed subspace of s. By Theorem 2.2, the surjection $A\,\cdot : \mathscr{E}(M; X)^q \to A \cdot \mathscr{E}(M; X)^q$ admits a continuous linear splitting (cf. [5, Thm. 0.1.1]).

In general, consider the following diagram in which the horizontal sequence is exact:

$$\begin{tikzcd}
 & & \Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r \arrow[dl, dashed, "\lambda"'] \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\Psi \arrow[r] & \mathscr{E}(N; Z)^q \arrow[r, "\Psi"] & \dfrac{\Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r}{B \cdot \mathscr{E}(M; X)^r} \arrow[r] & 0.
\end{tikzcd}$$

By Theorem 2.2, there is a continuous linear mapping $\lambda : \Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r \to \mathscr{E}(N; Z)^q$, making a commutative triangle, as shown.

Let $\mu : B \cdot \mathscr{E}(M; X)^r \to \mathscr{E}(M; X)^r$ denote a continuous linear splitting for the surjection $B\,\cdot : \mathscr{E}(M; X)^r \to B \cdot \mathscr{E}(M; X)^r$. If $f \in \Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r$, then $f - (\Phi \circ \lambda)(f) \in B \cdot \mathscr{E}(M; X)^r$; thus $f - (\Phi \circ \lambda)(f) = B \cdot \mu(f - (\Phi \circ \lambda)(f))$. Write $\nu(f) = \mu(f - (\Phi \circ \lambda)(f))$. Then

$$(\lambda, \nu) : \Phi\mathscr{E}(N; Z)^q + B \cdot \mathscr{E}(M; X)^r \to \mathscr{E}(N; Z)^q \oplus \mathscr{E}(M; X)^r$$

is a continuous linear splitting for the surjection (3.13). $\square$

§4. The diagram of initial exponents

Let $y = (y_1, \ldots, y_n)$ and let R be a submodule of $\mathbf{K}[[y]]^q$. Following Hironaka [6] (cf. [4, §1.4]), we associate to R a subset $\mathfrak{N}(R)$ of $\mathbf{N}^n \times \{1, \ldots, q\}$:

If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, put $|\beta| = \beta_1 + \cdots + \beta_n$. We order the $(n+2)$-tuples $(\beta_1, \ldots, \beta_n, j, |\beta|)$, where $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, lexicographically from the right. This induces a total ordering of $\mathbf{N}^n \times \{1, \ldots, q\}$.

Let $g \in \mathbf{K}[[y]]^q$. Write $g = \Sigma_{\beta,j} g_{\beta,j} y^{\beta,j}$, where $g_{\beta,j} \in \mathbf{K}$ and $y^{\beta,j}$ denotes the q-tuple $(0, \ldots, y^\beta, \ldots, 0)$ with $y^\beta = y_1^{\beta_1} \cdots y_n^{\beta_n}$ in the j'th place. Let $\operatorname{supp} g = \{(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\} : g_{\beta,j} \neq 0\}$ and let $\nu(g)$ denote the smallest element of $\operatorname{supp} g$. Let in g denote $g_{\nu(g)} y^{\nu(g)}$.

We define the *diagram of initial exponents* $\mathfrak{N}(R)$ as $\{\nu(g) : g \in R\}$. Clearly, $\mathfrak{N}(R) + \mathbf{N}^n = \mathfrak{N}(R)$, where addition is defined by $(\beta, j) + \gamma = (\beta + \gamma, j)$, $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, $\gamma \in \mathbf{N}^n$.

Put $\mathscr{D}(n, q) = \{\mathfrak{N} \subset \mathbf{N}^n \times \{1, \ldots, q\} : \mathfrak{N} + \mathbf{N}^n = \mathfrak{N}\}$. Let $\mathfrak{N} \in \mathscr{D}(n, q)$. Then there is a smallest finite subset $\mathfrak{B}$ of $\mathfrak{N}$ such that $\mathfrak{N} = \mathfrak{B} + \mathbf{N}^n$. We call $\mathfrak{B}$ the vertices of $\mathfrak{N}$.

The set $\mathscr{D}(n, q)$ is totally ordered as follows: To each $\mathfrak{N} \in \mathscr{D}(n, q)$, associate the sequence $\nu(\mathfrak{N})$ obtained by listing the vertices of $\mathfrak{N}$ in ascending order and completing this list to an infinite sequence by using ∞ for all the remaining terms. If $\mathfrak{N}^1, \mathfrak{N}^2 \in \mathscr{D}(n, q)$, we say that $\mathfrak{N}^1 < \mathfrak{N}^2$ provided that $\nu(\mathfrak{N}^1) < \nu(\mathfrak{N}^2)$ with respect to the lexicographic ordering on the set of such sequences.

Clearly, if $\mathfrak{N}^1 \supset \mathfrak{N}^2$, then $\mathfrak{N}^1 \leqq \mathfrak{N}^2$.

We now return to the context of the introduction. To the submodule $\mathscr{R}_{\mathbf{a}}$ of $\hat{\mathcal{O}}^q_{\varphi(\mathbf{a})}$, we associate a diagram $\mathfrak{N}_{\mathbf{a}}$ which depends on a choice of local coordinates $(y_1, \ldots, y_n)$ in a neighbourhood of $\varphi(\mathbf{a}) \in N$: Assume that N is an open subset of $\mathbf{K}^n$. If $b \in N$, then $\hat{\mathcal{O}}_b$ identifies with the ring of formal power series $\mathbf{K}[[y]]$ in the affine coordinates $y = (y_1, \ldots, y_n)$ of $\mathbf{K}^n$. Let $\mathbf{a} \in M^s_\varphi$. We put

$$\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}(\mathscr{R}_{\mathbf{a}}).$$

We say that the diagram of initial exponents $\mathfrak{N}_{\mathbf{a}}$ is *Zariski semicontinuous* on M^s_φ if, for every irreducible (germ of an) analytic subset X of M^s_φ, there exists a germ of a proper analytic subset Y of X such that (1) $\mathfrak{N}_{\mathbf{a}} = \mathfrak{N}_{\mathbf{b}}$ if $\mathbf{a}, \mathbf{b} \in X - Y$; (2) $\mathfrak{N}_{\mathbf{a}} < \mathfrak{N}_{\mathbf{b}}$ if $\mathbf{a} \in X - Y$, $\mathbf{b} \in Y$. (We use the same notation for a germ at a point and a representative of the germ in a suitable neighbourhood.)

THEOREM 4.1 (cf. [4, Thm. A]). *Assume that N is an open subset of $\mathbf{K}^n$. Then the Hilbert–Samuel function H_a is Zariski semicontinuous on M^s_φ if and only if the diagram of initial exponents $\mathfrak{N}_{\mathbf{a}}$ is Zariski semicontinuous on M^s_φ.*

§5. The formal division algorithm

We use the notation of Section 4. Let $\mathbf{K}$ be a field. Let $g^1, \ldots, g^t \in \mathbf{K}[[y]]^q$ and let $(\beta_i, j_i) = \nu(g^i)$, $i = 1, \ldots, t$. We associate to $g^1, \ldots, g^t$ the following decomposition of $\mathbf{N}^n \times \{1, \ldots, q\}$: Set $\Delta_0 = \varnothing$ and define $\Delta_i = ((\beta_i, j_i) + \mathbf{N}^n) - \bigcup_{k=0}^{i-1} \Delta_k$, $i = 1, \ldots, t$. Put $\Delta = \mathbf{N}^n \times \{1, \ldots, q\} - \bigcup_{i=1}^t \Delta_i$.

THEOREM 5.1 (Hironaka [6]; cf. [4, §6]). *Let $g^1, \ldots, g^t \in \mathbf{K}[[y]]^q$ and let $(\beta_i, j_i) = \nu(g^i)$, $i = 1, \ldots, t$. Then, for every $f \in \mathbf{K}[[y]]^q$, there exist unique $q_i \in \mathbf{K}[[y]]$, $i = 1, \ldots, t$ and $r \in \mathbf{K}[[y]]^q$ such that $(\beta_i, j_i) + \operatorname{supp} q_i \subset \Delta_i$, $i = 1, \ldots, t$, $\operatorname{supp} r \subset \Delta$, and*

$$f = \sum_{i=1}^{t} q_i g^i + r.$$

REMARK 5.2. Let A be an integral domain. Suppose that $\mathbf{K}$ is the field of fractions of A. Let $A[[y]]$ denote the subring of $\mathbf{K}[[y]]$ of formal power series with coefficients in A. Suppose that $g^1, \ldots, g^t \in A[[y]]^q$. Let S denote the multiplicative subset of A generated by the $g^i_{\beta_i, j_i}$, and let $S^{-1}A$ denote the subring of $\mathbf{K}$ comprising quotients with denominators in S. Then $S^{-1}A[[y]] \subset \mathbf{K}[[y]]$. In Theorem 5.1, if $f \in A[[y]]^q$, then $q_i \in S^{-1}A[[y]]$, $i = 1, \ldots, t$, and $r \in S^{-1}A[[y]]^q$ (cf. [4, Rmk. 6.5]). In fact, if A is any ring and each $g^i_{\beta_i, j_i} = 1$, the formal division algorithm applies to give quotients and remainder with coefficients in A.

REMARK 5.3. Suppose that $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and that f and $g^1, \ldots, g^t$ converge. Then the q_i and r all converge [6].

COROLLARY 5.4 (cf. [4, Cor. 6.8]). *Let R be a submodule of $\mathbf{K}[[y]]^q$. Let $\mathfrak{N} = \mathfrak{N}(R)$ be the diagram of initial exponents of R, and let (β_i, j_i), $i = 1, \ldots, t$, denote the vertices of $\mathfrak{N}$ (without repetitions). Choose $g^i \in R$ such that $\nu(g^i) = (\beta_i, j_i)$, $i = 1, \ldots, t$. Then:*

(1) *$\mathfrak{N} = \bigcup_{i=1}^t \Delta_i$, and $g^i, \ldots, g^t$ generate R.*

(2) *There is a unique set of generators $f^1, \ldots, f^t$ of R such that, for each i, $\operatorname{in} f^i = y^{\beta_i, j_i}$ and $\operatorname{supp}(f^i - y^{\beta_i, j_i}) \cap \mathfrak{N} = \varnothing$. If $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and R is generated by convergent elements, then each f^i converges.*

We call $f^1, \ldots, f^t$ in (2) the *standard basis* or *special generators* of R.

§6. Stratification and division

In this section, we will prove Theorem 3.12. We use the notation of the introduction. Suppose that φ is proper. Then, locally in N, there is a bound s on the number of distinct submodules $\mathscr{R}_a$ of $\hat{\mathcal{O}}_b^q$, where $a\in\varphi^{-1}(b)$ [4, Cor. 11.6]. Working locally, we assume that N is an open subset of $\mathbf{K}^n$ and we fix such an s. Thus, for all $b\in\varphi(M)$, there exists $\mathbf{a}\in\boldsymbol{\varphi}^{-1}(b)\subset M_\varphi^s$ such that $\mathscr{R}_{\mathbf{a}}=\bigcap_{a\in\varphi^{-1}(b)}\mathscr{R}_a$.

Suppose that the diagram of initial exponents $\mathfrak{N}_{\mathbf{a}}=\mathfrak{N}(\mathscr{R}_{\mathbf{a}})$ is Zariski semicontinuous on M_φ^s; i.e., there is a locally finite filtration of M_φ^s by closed analytic subsets,

$$M_\varphi^s=\Sigma_0\supset\Sigma_1\supset\cdots\supset\Sigma_\nu\supset\Sigma_{\nu+1}\supset\cdots,$$

such that, for all $\nu\in\mathbf{N}$, $\mathfrak{N}_{\mathbf{a}}$ is constant on $\Sigma_\nu-\Sigma_{\nu+1}$ and $\mathfrak{N}_{\mathbf{a}}<\mathfrak{N}_{\mathbf{a}'}$, if $\mathbf{a}\in\Sigma_\nu-\Sigma_{\nu+1}$, $\mathbf{a}'\in\Sigma_{\nu+1}$. It follows that, for all $\mathbf{a}\in\Sigma_\nu-\boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(\Sigma_{\nu+1}))$, $\mathscr{R}_{\mathbf{a}}=\bigcap_{a\in\varphi^{-1}(\boldsymbol{\varphi}(\mathbf{a}))}\mathscr{R}_a$.

Therefore, there is a locally finite partition $\{X_\mu\}_{\mu\in\mathbf{N}}$ of M_φ^s such that, for each μ:

(6.1) X_μ is a relatively compact connected smooth semianalytic subset of M_φ^s, and $\bar{X}_\mu$ lies in a product coordinate chart U_μ in M^s.

(6.2) $\bar{X}_\mu-X_\mu\subset\bigcup_{\lambda<\mu}X_\lambda$.

(6.3) $\mathfrak{N}_{\mathbf{a}}$ is constant, say $\mathfrak{N}_{\mathbf{a}}=\mathfrak{N}_\mu$, on X_μ.

(6.4) Let $Y_\mu=\boldsymbol{\varphi}(\bigcup_{\lambda\le\mu}X_\lambda)$. Then, for all $\mathbf{a}\in X_\mu-\boldsymbol{\varphi}^{-1}(Y_{\mu-1})$, $\mathscr{R}_{\mathbf{a}}=\bigcap_{a\in\varphi^{-1}(\boldsymbol{\varphi}(\mathbf{a}))}\mathscr{R}_a$.

We can, of course, assume:

(6.5) $\boldsymbol{\varphi}\mid X_\mu$ has constant rank.

Let $\mu\in\mathbf{N}$. For each $\mathbf{a}\in X_\mu$, let $G_{\mathbf{a}}^i(y)=y^{\beta_i,j_i}-r_{\mathbf{a}}^i(y)$, $i=1,\ldots,t$, denote the standard basis of $\mathscr{R}_{\mathbf{a}}$, where, for each i, in $G_{\mathbf{a}}^i=y^{\beta_i,j_i}$ and $\operatorname{supp} r_{\mathbf{a}}^i\cap\mathfrak{N}_{\mathbf{a}}=\varnothing$. Write $r_{\mathbf{a}}^i(y)=\sum_{\beta,j}r_{\beta,j}^i(\mathbf{a})y^{\beta,j}$. By [4, Thm. B], we can also assume:

(6.6) For each $i=1,\ldots,t$ and $(\beta,j)\in\mathbf{N}^n\times\{1,\ldots,q\}$, $r_{\beta,j}^i\in\mathscr{M}(X_\mu)$.

The notation above will be fixed throughout this section. Let Z be a closed subanalytic subset of N. For each $\mu\in\mathbf{N}$, let $Z_\mu=Z\cap Y_\mu$. We will prove:

THEOREM 6.7. *Let $\mathbf{K}=\mathbf{R}$. Then, for each μ, $\mathscr{R}(Y_\mu;Y_{\mu-1}\cup Z_\mu)$ is a quotient of s.*

Theorem 3.12 then follows:

PROOF OF THEOREM 3.12. Using a partition of unity on N, we can reduce to the local situation above. By Corollary 3.11, for each μ, the following sequence is exact:

$$0 \to \mathscr{R}(Y_\mu; Y_{\mu-1} \cup Z_\mu) \to \mathscr{R}(Y_\mu; Z_\mu) \to \mathscr{R}(Y_{\mu-1}; Z_{\mu-1}) \to 0.$$

The result follows by induction on μ, using Theorem 6.7 and Corollary 2.3. □

If W is a locally closed subset of N, set $J(W) = \Pi_{b\in W}\hat{\mathcal{O}}_b$. If $G \in J(W)$, we write $G = (G_b)$, where $G_b \in \hat{\mathcal{O}}_b$, $b \in W$. There is a canonical injection $\mathscr{E}(W) \to J(W)$.

DEFINITION AND REMARKS 6.8. Let V be an open subset of $\mathbf{K}^n$, and let W be an immersed analytic submanifold of V. Let $G \in J(W)$; say $G = (G_b)$, where $G_b(y) = \Sigma_\beta g_\beta(b) y^\beta \in \hat{\mathcal{O}}_b$, $b \in W$. Suppose that each $g_\beta \in \mathcal{O}(W)$, where $\mathcal{O}(W)$ denotes the ring of analytic functions on W (i.e., restrictions of analytic functions defined in neighborhoods of W).

Regarding both $b = (b_1, \ldots, b_n)$ and $y = (y_1, \ldots, y_n)$ as variables in V, if $\eta = (\eta_1, \ldots, \eta_n) \in \mathbf{K}^n$, we write

$$D_{\eta,b} = \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial b_j} \quad \text{and} \quad D_{\eta,y} = \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial y_j};$$

$D_{\eta,b}$ and $D_{\eta,y}$ are the directional derivatives in the direction η. If $b \in W$ and η is tangent to (one of the branches of) W at b, then the directional derivative $D_{\eta,b}G_b(y)$ at b is well-defined, as is the (formal) directional derivative $D_{\eta,y}G_b(y)$.

We say that $G \in \tilde{\mathcal{O}}(W)$ if, for each $b \in W$ and each tangent η to W at b,

$$D_{\eta,b}G_b(y) = D_{\eta,y}G_b(y). \tag{6.9}$$

If $G \in \tilde{\mathcal{O}}(W)$, we say that G is *analytic with respect to the variables of W and formal with respect to the normal variables*.

This definition can be justified as follows: Let Σ denote a local branch of W. Choose (analytic) local coordinates $(u, v) = (u_1, \ldots, u_k, v_1, \ldots, v_{n-k})$ of V such that Σ is given by $v = 0$. Write G_b as

$$G_b(u, v) = \sum_{\beta\in\mathbf{N}^{n-k}} \left(\sum_{\alpha\in\mathbf{N}^k} \gamma_{\alpha,\beta}(b) \frac{u^\alpha}{\alpha!} \right) \frac{v^\beta}{\beta!},$$

where the $\gamma_{\alpha,\beta} \in \mathcal{O}(\Sigma)$. Then (6.9) implies that $\Sigma_\alpha \gamma_{\alpha,\beta}(b) u^\alpha/\alpha!$ is the Taylor expansion at b of the analytic function $\gamma_{0,\beta}$.

If $\mathbf{K} = \mathbf{R}$ and $G \in \tilde{\mathcal{O}}(W)$, it follows from E. Borel's lemma [22, IV.3.5] and Łojasiewicz's glueing theorem [11, I.5.5] that $G \in \mathcal{E}(W)$.

We return to the stratification introduced above. Let $\mu \in \mathbf{N}$. For each $i = 1, \ldots, t$ and each $b \in Y_\mu - Y_{\mu-1}$, let $G_b^i(y) = G_{\mathbf{a}}^i(y)$, where $\mathbf{a} \in X_\mu$ and $\varphi(\mathbf{a}) = b$. (The G_b^i are well-defined, by (6.4).) Write $G_b^i(y) = y^{\beta_i, j_i} - \Sigma_{\beta,j} r_{\beta,j}^i(b) y^{\beta,j}$, where $r_{\beta,j}^i(b) = 0$ if $(\beta, j) \in \mathfrak{N}_\mu$. Put $G^i = (G_b^i)_{b \in Y_\mu - Y_{\mu-1}} \in J(Y_\mu - Y_{\mu-1})^q$. By (6.5), $Y_\mu - Y_{\mu-1}$ is an immersed analytic submanifold of $N - Y_{\mu-1}$. The following proposition together with (6.1)–(6.6) establish Theorem 1.3.

PROPOSITION 6.10. *For each $i = 1, \ldots, t$, $G^i \in \tilde{\mathcal{O}}(Y_\mu - Y_{\mu-1})^q$.*

PROOF. Let $b_0 \in Y_\mu - Y_{\mu-1}$ and let Σ denote one of the local branches of $Y_\mu - Y_{\mu-1}$ at b_0. By (6.5), there exists $\mathbf{a}_0 \in X_\mu - \varphi^{-1}(Y_{\mu-1})$ and (a germ at $\mathbf{a}_0$ of) an analytic submanifold P of X_μ such that φ induces an analytic diffeomorphism of P onto Σ.

LEMMA 6.11. *There is a proper closed analytic subset Q of P such that, for each $i = 1, \ldots, t$ and each $\mathbf{a} = (a^1, \ldots, a^s) \in P - Q$, there exist $H_{i,\mathbf{a}}^j \in \hat{\mathcal{O}}_{a^j}^r$, $j = 1, \ldots, s$, satisfying*:

(1) $\hat{A}_{a^j} \cdot (G_{\varphi(\mathbf{a})}^i \circ \hat{\varphi}_{a^j}) + \hat{B}_{a^j} \cdot H_{i,\mathbf{a}}^j = 0.$

(2) *The coefficients of $H_{i,\mathbf{a}}^j$ are analytic functions of $\mathbf{a} \in P - Q$.*

PROOF. The coordinate chart U_μ of (6.1) is a product $U_\mu = \Pi_{j=1}^s U_\mu^j$, where each U_μ^j is a coordinate chart in M. We use $x = (x_1, \ldots, x_m)$ for local coordinates in any of the U_μ^j. Let $B_{\mathbf{a}}^j(x) = \hat{B}_{a^j}(x)$, $j = 1, \ldots, s$, where $\mathbf{a} = (a^1, \ldots, a^s) \in P \subset U_\mu$. The coefficients of the $B_{\mathbf{a}}^j(x)$ are analytic functions on P.

For each $\mathbf{a} \in P$, there is an evaluation mapping $h \mapsto h(\mathbf{a})$ of $\mathcal{O}(P)$ onto $\mathbf{K}$. If $f = \Sigma_{\alpha,l} f_{\alpha,l} x^{\alpha,l} \in \mathcal{O}(P)[[x]]^p$, we write $f(\mathbf{a}; x) = \Sigma f_{\alpha,l}(\mathbf{a}) x^{\alpha,l}$ when the coefficients are evaluated at $\mathbf{a} \in P$. Let $\nu(f) \in \mathbf{N}^m \times \{1, \ldots, p\}$ denote the smallest (α, l) such that $f_{\alpha,l} \in \mathcal{O}(P)$ is nonzero.

For each $j = 1, \ldots, s$, let $\mathcal{B}^j$ denote the submodule of $\mathcal{O}(P)[[x]]^p$ generated by the columns of $B_\cdot^j(x)$. Put $\mathfrak{N}^j = \{\nu(f) : f \in \mathcal{B}^j\}$. $\mathfrak{N}^j + \mathbf{N}^m = \mathfrak{N}^j$. If $\mathbf{a} \in P$, let $\mathcal{B}_{\mathbf{a}}^j$ denote the submodule $\hat{B}_{a^j} \cdot \mathbf{K}[[x]]^r$ of $\mathbf{K}[[x]]^p$; i.e., the submodule generated by the columns of $B_{\mathbf{a}}^j(x)$ (evaluated at $\mathbf{a}$). Put $\mathfrak{N}_{\mathbf{a}}^j = \mathfrak{N}(\mathcal{B}_{\mathbf{a}}^j)$. Let (α_k^j, l_k^j), $k = 1, \ldots, t(j)$, denote the vertices of $\mathfrak{N}^j$. Then, for each $j = 1, \ldots, s$, there exists $\psi_k^j \in \mathcal{B}^j$, $k = 1, \ldots, t(j)$, such that $\nu(\psi_k^j) = (\alpha_k^j, l_k^j)$. Write $\psi_k^j = \Sigma \psi_{k,\alpha,l}^j x^{\alpha,l}$, where each $\psi_{k,\alpha,l}^j \in \mathcal{O}(P)$. Let $Q_k^j = \{\mathbf{a} \in P : \psi_{k,\alpha,l}^j(\mathbf{a}) = 0$, where $(\alpha,l) = (\alpha_k^j, l_k^j)\}$. Put $Q = \bigcup_{j=1}^s \bigcup_{k=1}^{t(j)} Q_k^j$.

Let $\mathbf{a}\in P-Q$. Then each $\psi_k^j(\mathbf{a};x)\in\mathscr{B}_{\mathbf{a}}^j$ and $\nu(\psi_k^j(\mathbf{a};x))=(\alpha_k^j,l_k^j)$; thus each $\mathfrak{N}^j\subset\mathfrak{N}_{\mathbf{a}}^j$. In fact, it is not difficult to see that $\mathfrak{N}_{\mathbf{a}}^j=\mathfrak{N}^j$ (cf. [4, Lemma 7.1]).

Let $\mathbf{a}\in P-Q$. Write $F_{i,\mathbf{a}}^j(x)=-\hat{A}_{a^j}(x)\cdot(G_{\varphi(\mathbf{a})}^i\circ\hat{\varphi}_{a^j})(x)$, $i=1,\dots,t$, $j=1,\dots,s$. Let $\{\Delta_k^j,\Delta^j\}_{1\le k\le t(j)}$ denote the decomposition of $\mathbf{N}^m\times\{1,\dots,p\}$ determined by the vertices (α_k^j,l_k^j) of $\mathfrak{N}^j$, as in Section 5. By Theorem 5.1, since $F_{i,\mathbf{a}}^j\in\mathscr{B}_{\mathbf{a}}^j$, there exist unique $\xi_{ik,\mathbf{a}}^j\in\mathbf{K}[[x]]$, $k=1,\dots,t(j)$, such that

$$(\alpha_k^j,l_k^j)+\operatorname{supp}\xi_{ik,\mathbf{a}}^j\subset\Delta_k^j\quad\text{and}\quad F_{i,\mathbf{a}}^j=\sum_{k=1}^{t(j)}\xi_{ik,\mathbf{a}}^j\cdot\psi_{k,\mathbf{a}}^j,$$

where $\psi_{k,\mathbf{a}}^j(x)=\psi_k^j(\mathbf{a};x)$. According to Remark 5.2, the coefficients of the $\xi_{ik,\mathbf{a}}^j(x)$ are analytic functions of $\mathbf{a}\in P-Q$. The lemma clearly follows from our choice of ψ_k^j above. □

We now complete the proof of Proposition 6.10: Let $b\in\Sigma$ and let $\eta=(\eta_1,\dots,\eta_n)$ be a tangent vector to Σ at b. Say $b=\varphi(\mathbf{a})$, where $\mathbf{a}\in P$. Let ξ be the tangent vector to $P\subset U_\mu$ at $\mathbf{a}$ such that $\eta=d\varphi(\mathbf{a})\cdot\xi$ (where $d\varphi(\mathbf{a})$ denotes the tangent mapping of $\varphi\mid P$ at $\mathbf{a}$).

First assume that $\mathbf{a}\in P-Q$. We write $\nabla_bG_b^i$ or $\nabla_yG_b^i$ to denote the gradient of $G_b^i(Y)$ with respect to the variables of Σ or with respect to y, respectively. Then, for each $i=1,\dots,t$ and each $j=1,\dots,s$,

$$\begin{aligned}
&D_{\xi,\mathbf{a}}(G_{\varphi(\mathbf{a})}^i\circ\hat{\varphi}_{a^j})\\
&\quad=((\nabla_bG_b^i)\circ\hat{\varphi}_{a^j})\cdot D_{\xi,\mathbf{a}}\varphi(\mathbf{a})+((\nabla_yG_b^i)\circ\hat{\varphi}_{a^j})\cdot D_{\xi,\mathbf{a}}(\hat{\varphi}_{a^j}(x^j)-\varphi(\mathbf{a}))\\
&\quad=(D_{\eta,b}G_b^i)\circ\hat{\varphi}_{a^j}-(D_{\eta,y}G_b^i)\circ\hat{\varphi}_{a^j}+((\nabla_yG_b^i)\circ\hat{\varphi}_{a^j})\cdot D_{\xi,\mathbf{a}}\hat{\varphi}_{a^j}(x^j),
\end{aligned}$$

and

$$D_{\xi,\mathbf{x}}(G_{\varphi(\mathbf{a})}^i\circ\hat{\varphi}_{a^j})=((\nabla_yG_b^i)\circ\hat{\varphi}_{a^j})\cdot D_{\xi,\mathbf{x}}\hat{\varphi}_{a^j}(x^j),$$

where $\mathbf{x}=(x^1,\dots,x^s)$ denotes the coordinates of U_μ. Since φ is analytic, then $D_{\xi,\mathbf{a}}\hat{\varphi}_{a^j}=D_{\xi,\mathbf{x}}\hat{\varphi}_{a^j}$ ("Taylor expansion commutes with differentiation"); thus,

$$(D_{\xi,\mathbf{a}}-D_{\xi,\mathbf{x}})(G_{\varphi(\mathbf{a})}^i\circ\hat{\varphi}_{a^j})=((D_{\eta,b}-D_{\eta,y})G_b^i)\circ\hat{\varphi}_{a^j}.$$

Likewise, since the entries of the matrices A and B are analytic, then $D_{\xi,\mathbf{a}}-D_{\xi,\mathbf{x}}$ vanishes when applied to $\hat{A}_{a^j}(x^j)$ and $\hat{B}_{a^j}(x^j)$. It follows from Lemma 6.11 that, for each $i=1,\dots,t$ and $j=1,\dots,s$,

$$\hat{A}_{a^j}\cdot((D_{\eta,b}-D_{\eta,y})G_b^i)\circ\hat{\varphi}_{a^j}\in\operatorname{Im}\hat{B}_{a^j};$$

i.e., $(D_{\eta,b} - D_{\eta,y})G_b^i \in \mathcal{R}_{\mathbf{a}}$. Since $\mathrm{supp}(D_{\eta,b} - D_{\eta,y})G_b^i \cap \mathfrak{N}_\mu = \emptyset$ (where supp is with respect to y), then $(D_{\eta,b} - D_{\eta,y})G_b^i = 0$.

For arbitrary $\mathbf{a} \in P$, extend η to an analytic vector field tangent to Σ in a neighborhood of $b = \varphi(\mathbf{a})$. Since the coefficients of $D_{\eta,b}G_b^i(y)$ and $D_{\eta,y}G_b^i(y)$ are analytic functions on Σ and $P - Q$ is adherent to $\mathbf{a}$, it follows from the preceding paragraph that again $(D_{\eta,b} - D_{\eta,y})G_b^i = 0$, as required. □

We will now prove a theorem (6.15 below) on division by the special generators G^i of Proposition 6.10, from which Theorem 6.7 will immediately follow.

LEMMA 6.12 ("Hestenes's lemma for subanalytic sets"). *Let V be an open subset of $\mathbf{R}^n$, and let $Z \subset Y$ denote closed subsets of V, where Y is subanalytic. Let $H \in J(Y)$; say $H = (H_b)$, where $H_b(y) = \Sigma_\beta h_\beta(b) y^\beta \in \hat{\mathcal{O}}_b$, $b \in Y$. Suppose that $H \mid (Y - Z) \in \mathscr{E}(Y - Z)$ and that each h_β is continuous on Y and zero on Z. Then $H \in \mathscr{E}(Y)$.*

PROOF. See [3, Cor. 8.2]. □

Let V be an open subset of $\mathbf{R}^n$, and let $Z \subset Y$ denote closed subsets of V. Let $\mathscr{D}$ denote the Fréchet space $\mathscr{E}(V; Z)/(\mathscr{C}^\infty(V; Y) \cap \mathscr{E}(V; Z))$ of restrictions to Y of $\mathscr{C}^\infty$ functions which are flat on Z. There is a continuous linear injection of $\mathscr{D}$ into the space $\mathscr{C}^0(Y; Z)$ of continuous functions on Y which vanish on Z.

LEMMA 6.13. *Suppose that Z is subanalytic. Let $\tau \in \mathscr{M}(Y - Z)$. If $f \in \mathscr{D}$, then $\tau \cdot f$ extends in a unique way to an element of $\mathscr{D}$ (also denoted $\tau \cdot f$). Moreover, the mapping $f \mapsto \tau \cdot f$ of $\mathscr{D}$ to itself is continuous and linear.*

PROOF. The assertion is local, so we can assume there are analytic functions f_i, g_i, $i = 1, \ldots, r$, defined on V such that $Y \cap \bigcap_{i=1}^r g_i^{-1}(0) \subset Z$ and, for each i, $\tau \cdot g_i = f_i$ on $Y - Z$. Put $P_i = Z \cup g_i^{-1}(0)$, $i = 1, \ldots, r$, and $P = \bigcap P_i$, so that $P \cap Y = Z$. Let $\pi : \mathscr{E}(V; P) \to \mathscr{D}$ be the canonical surjection.

The mapping $(F_i) \mapsto \Sigma F_i$ from $\bigoplus_{i=1}^r \mathscr{E}(V; P_i)$ to $\mathscr{E}(V; P)$ is surjective, as follows: Let $F \in \mathscr{E}(V; P)$. By Lojasiewicz's glueing theorem [11, I.5.5] and Whitney's extension theorem [11, I.4.1], there exists $F_r \in \mathscr{E}(V; P_r)$ such that $F_r = F$ on $P_1 \cap \cdots \cap P_{r-1}$. Then $F = (F - F_r) + F_r$, where $F - F_r \in \mathscr{E}(V; P_1 \cap \cdots \cap P_{r-1})$. The assertion follows by induction. Let ρ: $\bigoplus_{i=1}^r \mathscr{E}(V; P_i) \to \mathscr{D}$ denote the induced surjection.

For each $i = 1, \ldots, r$, write $\tau_i = f_i/g_i \in \mathscr{M}(V - P_i)$. It follows from Łojasiewicz's inequality [11, IV.4.1] that, if $G \in \mathscr{E}(V; P_i)$, then $\tau_i \cdot G$ extends in a unique way to an element of $\mathscr{E}(V; P_i)$ (also written $\tau_i \cdot G$), and that the induced mapping $G \mapsto \tau_i \cdot G$ of $\mathscr{E}(V; P_i)$ to itself is continuous and linear [22, IV.4.2]. Define S: $\bigoplus_{i=1}^r \mathscr{E}(V; P_i) \to \mathscr{E}(V; P)$ by $S((F_i)) = \Sigma\, \tau_i \cdot F_i$, where $(F_i) \in \bigoplus_{i=1}^r \mathscr{E}(V; P_i)$.

For each i, $\tau = \tau_i$ on $Y - P_i$. It follows that there is a unique continuous linear mapping T from $\mathscr{D}$ to itself such that $T \circ p = \pi \circ S$; clearly $T(f) = \tau \cdot f$, where $f \in \mathscr{D}$. □

Assume that $\mathbf{K} = \mathbf{R}$. If $H = (H_1, \ldots, H_t) \in \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^t$, define $\Gamma(H) \in J(Y_\mu)^q$ by

$$\Gamma(H)_b = \sum_{i=1}^t H_{i,b} \cdot G_b^i, \qquad b \in Y_\mu - (Y_{\mu-1} \cup Z_\mu),$$

$$\Gamma(H)_b = 0, \qquad b \in Y_{\mu-1} \cup Z_\mu.$$

Lemma 6.14. (1) *Let* $H \in \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^t$. *Then* $\Gamma(H) \in \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^q$.

(2) *The mapping* $\Gamma: \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^t \to \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^q$ *thus defined is continuous and linear.*

Proof. (1) Let $b \in Y_\mu - Y_{\mu-1}$. Then $G_b^i(y) = y^{\beta_i, j_i} - \Sigma_{\beta,j}\, r_{\beta,j}^i(b) y^{\beta,j}$, where each $r_{\beta,j}^i \circ (\varphi \mid (X_\mu - \varphi^{-1}(Y_{\mu-1})))$ is the restriction of an element on $\mathscr{M}(X_\mu)$, and $H_{i,b}(y) = \Sigma_\beta\, h_{i,\beta}(b) y^\beta$, where each $h_{i,\beta}$ is the restriction of a $\mathscr{C}^\infty$ function flat on $Y_{\mu-1} \cup Z_\mu$. For each $b \in Y_\mu$, write $\Gamma(H)_b(y) = \Sigma_{\beta,j}\, \gamma_{\beta,j}(b) y^{\beta,j}$. It follows from Lemma 6.13 that each $\gamma_{\beta,j} \circ \varphi$, defined on $\bigcup_{\lambda \leq \mu} X_\lambda$, is the restriction of a $\mathscr{C}^\infty$ function which is flat on $\varphi^{-1}(Y_{\mu-1} \cup Z_\mu) \supset \bigcup_{\lambda < \mu} X_\lambda$. Since $\varphi \mid \bigcup_{\lambda \leq \mu} X_\lambda$ is proper, each $\gamma_{\beta,j}$ is a continuous function on Y_μ which vanishes on $Y_{\mu-1} \cup Z_\mu$. By Proposition 6.10, $\Gamma(H) \mid (Y_\mu - Y_{\mu-1}) \in \mathscr{E}(Y_\mu - Y_{\mu-1})^q$. It follows from Lemma 6.12 that $\Gamma(H) \in \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^q$.

(2) Since Y_μ is subanalytic, the topology of $\mathscr{E}(Y_\mu)$ is defined by the seminorms

$$\| G \|_l^K = \sup_{\substack{b \in K \\ |\beta| \leq l}} \left| \frac{\partial^{|\beta|} G}{\partial y^\beta}(b) \right|,$$

where $K \subset Y_\mu$ is compact, $l \in \mathbf{N}$ and $G \in \mathscr{E}(Y_\mu)$ (cf. [9; 22, IV.3.11]). The assertion follows from Lemma 6.13, Leibniz's formula, and the fact that $\varphi \mid \bigcup_{\lambda \leq \mu} X_\lambda$ is proper. □

Let $(\operatorname{Im} \Gamma)^\wedge$ denote the elements of $\mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^q$ which formally belong to the image of the mapping Γ of Lemma 6.14(2); i.e., $(\operatorname{Im}\Gamma)^\wedge = \{G \in \mathcal{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^q$: for all $b \in Y_\mu - (Y_{\mu-1} \cup Z_\mu)$, there exists $H_{i,b} \in \hat{\mathcal{O}}_b$, $i = 1, \ldots, t$, such that $\hat{G}_b = \Sigma H_{i,b} \cdot G_b^i\}$. Clearly, $(\operatorname{Im}\Gamma)^\wedge$ is a closed subspace of $\mathcal{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)^q$ containing $\operatorname{Im}\Gamma$.

THEOREM 6.15. $\operatorname{Im}\Gamma = (\operatorname{Im}\Gamma)^\wedge$.

PROOF OF THEOREM 6.7. It follows from (6.4) that $\mathcal{R}(Y_\mu; Y_{\mu-1} \cup Z_\mu) = (\operatorname{Im}\Gamma)^\wedge$. Then $\mathcal{R}(Y_\mu; Y_{\mu-1} \cup Z_\mu)$ is a quotient of s, by Theorems 2.5 and 6.15. $\square$

PROOF OF THEOREM 6.15. Suppose that $G \in (\operatorname{Im}\Gamma)^\wedge$. Let $b \in Y_\mu - (Y_{\mu-1} \cup Z_\mu)$ and let $\mathbf{a} \in X_\mu$ such that $\varphi(\mathbf{a}) = b$. Then $\hat{G}_b = \mathcal{R}_{\mathbf{a}} = \bigcap_{a \in \varphi^{-1}(b)} \mathcal{R}_a$. Let $\{\Delta_i, \Delta\}$ be the partition of $\mathbf{N}^n \times \{1, \ldots, q\}$ associated to the initial exponents (β_i, j_i) of the G_b^i, as in Section 5. By the formal division algorithm (Theorem 5.1), there are unique $H_{i,b} \in \hat{\mathcal{O}}_b$, $i = 1, \ldots, t$, such that $(\beta_i, j_i) + \operatorname{supp} H_{i,b} \subset \Delta_i$ and

$$\hat{G}_b = \sum_{i=1}^{t} H_{i,b} \cdot G_b^i. \tag{6.16}$$

Put $H_{i,b} = 0$ if $b \in Y_{\mu-1} \cup Z_\mu$. Define $H_i \in J(Y_\mu)$ by $H_i = (H_{i,b})_{b \in Y_\mu}$. We claim that $H_i \in \mathcal{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)$, $i = 1, \ldots, t$:

Write $H_{i,b} = \Sigma_\beta H_{i,\beta}(b) y^\beta$. By the formal division algorithm (cf. Remark 5.2) and Lemma 6.13, each $H_{i,\beta} \circ (\varphi \mid \bigcup_{\lambda \leq \mu} X_\lambda)$ is the restriction of a $\mathcal{C}^\infty$ function which is flat on $\varphi^{-1}(Y_{\mu-1} \cup Z_\mu)$. Since $\varphi \mid \bigcup_{\lambda \leq \mu} X_\lambda$ is proper, each $H_{i,\beta}$ is continuous on Y_μ and vanishes on $Y_{\mu-1} \cup Z_\mu$. (By (6.5), each $H_{i,\beta}$ is, in fact, $\mathcal{C}^\infty$ on the immersed submanifold $Y_\mu - Y_{\mu-1}$ of $N - Y_{\mu-1}$.)

Let $b \in Y_\mu - Y_{\mu-1}$. Suppose that $\eta = (\eta_1, \ldots, \eta_n) \in \mathbf{R}^n$ is tangent to $Y_\mu - Y_{\mu-1}$ at b. Since G and the G^i are $\mathcal{C}^\infty$ Whitney fields, then (in the notation of 6.8) we have:

$$\begin{aligned} D_{\eta,b}\hat{G}_b(y) &= D_{\eta,y}\hat{G}_b(y), \\ D_{\eta,b}G_b^i(y) &= D_{\eta,y}G_b^i(y), \end{aligned} \tag{6.17}$$

$i = 1, \ldots, q$. By (6.16) and (6.17),

$$\sum_{i=1}^{t} (D_{\eta,b}H_{i,b} - D_{\eta,y}H_{i,b}) \cdot G_b^i = 0.$$

But, for each i, $(\beta_i, j_i) + \operatorname{supp}(D_{\eta,b}H_{i,b} - D_{\eta,y}H_{i,b}) \subset \Delta_i$ (where supp is with respect to y). Therefore, by the uniqueness of formal division, $D_{\eta,b}H_{i,b} = D_{\eta,y}H_{i,b}$, $i = 1, \ldots, t$. Thus each H_i restricts to an element of $\mathscr{E}(Y_\mu - Y_{\mu-1})$ (cf. Remarks 6.8). It follows from Lemma 6.12 that each $H_i \in \mathscr{E}(Y_\mu; Y_{\mu-1} \cup Z_\mu)$, as required. □

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